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Growing Cayley trees described by a Fermi distribution

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In growing Cayley trees with thermal noise we show that the distribution of the bond strengths (energies) is described by the Fermi statistics. The evolution of these hierarchical networks reduces to the Eden model and the invasion percolation model in the limits $T\to 0$ and $T\to\infty$, respectively. We discuss the relation of the present results with the scale-free networks described by Bose statistics.

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I. INTRODUCTION

Recently it has been shown that the Bose statistics [1,2] can be used to describe a scale-free network [3,4] with fitness of the nodes [5]. Since scale-free networks are continuously growing and develop a power-law connectivity distribution, it is interesting to investigate their relation with self-organized processes [6–8]. In order to address this problem we present a model of invasion percolation with temperature defined on a Cayley tree, self-organized in the limit $T\to 0$. Our results show that the model can be solved analytically using the same technique used in the case of the scale-free networks described by the Bose statistics.

The invasion percolation model [6] is the most famous and simple example of evolution with quenched disorder. It describes the displacement of a fluid in a porous medium. The porous medium is given by a random network constituted by bonds with different strengths p chosen with a uniform probability in the interval (0,1). The classical asymptotic structure generated in this way is a fractal and the distribution of the strength values at the interface converges in time to a step function $\theta(p-p_c)$, where p_c is the percolation threshold of the static percolation problem. In order to include the effect of fluctuations on the dynamics of invasion, present in a real stochastic cases, we include a temperaturelike noise T [9–12]. A structure in which the invasion percolation dynamics can be defined is a Cayley tree [13,14], also known as the Bethe lattice. In this structure there are no loops, and the number of nodes in the bulk are of the same order of magnitude as the nodes at the interface. Therefore a Cayley tree is considered to be a good representation of a $d=\infty$ space and it is used in mean field calculations [15] and in the study of branching processes [16]. In this work we find that the distribution of bond strengths at the interface is no more a step function but is described by the Fermi distribution with temperature T where the bond strength plays the role of energy. By comparison with scale-free networks following the Bose statistics I show that both networks grow continuously in time: in the power-law network at each time a node is connected to the network by m links while in the Cayley tree model, at each time, a node grows giving rise to m new nodes (see Fig. 1). The dynamics of the two networks change in time awarding with a new link the fitter nodes in the power-law network or choosing the less fit nodes to grow in the Cayley tree network. At the same time, in the power-law network, the distribution of the energies of the chosen nodes converges to a Bose distribution while in a Cayley tree model the distribution of the energies at the interface converges to a Fermi distribution.

II. THE MODEL

The Cayley tree (or Bethe lattice) is a loop-free network in which there are three classes of nodes: *the root node*, which is at the origin of the tree and has connectivity m, the *nodes at the interface* with connectivity 1; and the *nodes in the bulk* (below the interface) with connectivity $m+1$.

We start from the root of the tree (node $i=1$) and we link it to m new nodes $i=2,3,\ldots,m+1$. We indicate each node with a subsequent number, t_i indicating the time in which it arrives in the interface (see Fig. 2).

At each time step we choose one node to grow, giving rise to m new nodes. Consequently, the interface of the tree grows linearly in time, and the growing node is chosen at each time from the growing number of active ones. In order to mimic the quenched noise of the medium we assign to

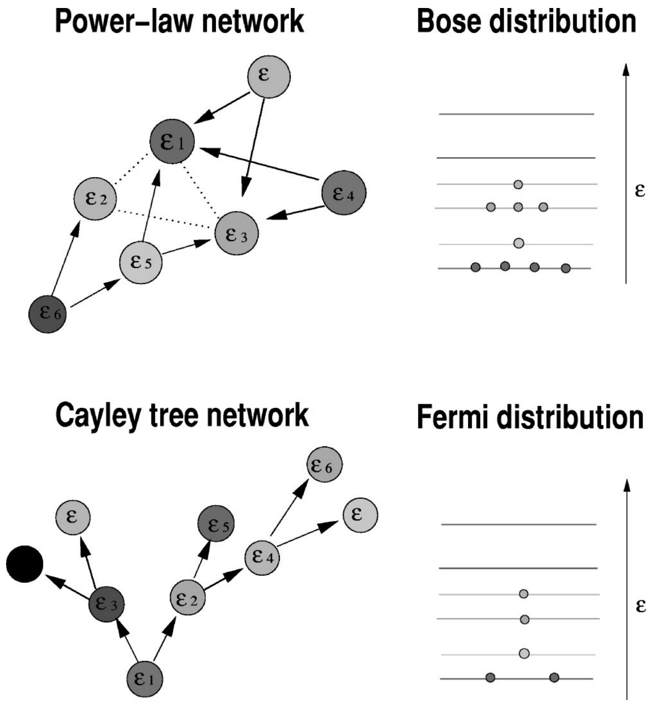

FIG. 1. Symmetric construction of a power-law network and the Cayley tree model considered in this paper.

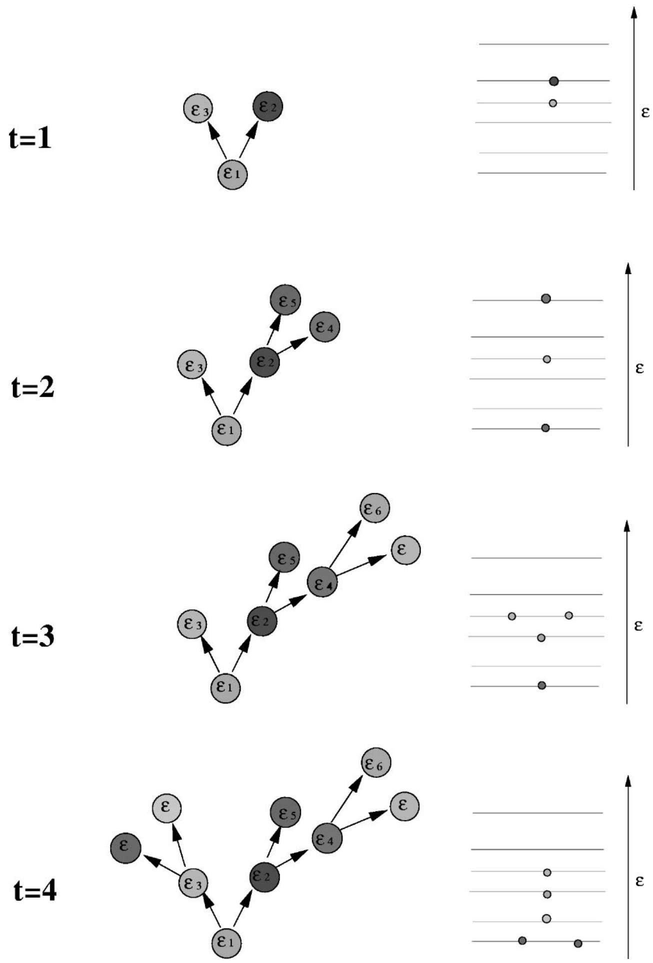

FIG. 2. Description of the dynamics of the model with $m=2$. At time $t=1$ the root node $i=1$ with energy ϵ_1 grows giving rise to m new nodes $i=2,3$ with energies ϵ_2,ϵ_3. Node $i=1$ is in the bulk or below the interface (connectivity m) while the nodes $i=2,3$ have connectivity 1 and are at the interface. At time $t=2$ the node $i=2$ is chosen to grow, it leaves the interface giving rise to m new nodes, $i=4,5$. At time $t=3,4$ nodes $i=4$ and $i=3$ are chosen to grow. On the right-hand side of the figure we plot the density of states of the node in the interface. For example, at time $t=4$ the nodes at the interface are $i=5,6,7,8,9$ and we can draw the density of states of the node at the interface by placing a particle in the energy levels $\epsilon_5=\epsilon_8$, $\epsilon_9<\epsilon_6<\epsilon_7$ as indicated in the picture.

each node of the tree an energy ϵ from a fixed random distribution $p(\epsilon)$.

We assume that higher energy nodes are more likely to grow than lower energy ones and that the probability Π_i for the active node i (with energy ϵ_i) to grow at time t is given by

$$\Pi_i=\frac{e^{\beta\epsilon_i}}{\sum_{j\in Int(t)} e^{\beta\epsilon_j}}, \tag{1}$$

where the sum in the denominator is extended to all nodes j that belong to the interface Int(t) at time t. The model depends on the parameter β. Tuning β we change the nature of the model and the spatial aspect of the tree. In the $\beta\to 0$ limit, high and low energy nodes are equally probable to grow and the model reduces to the *Eden model* while in the $\beta\to\infty$ limit the dynamics becomes extremal such that only the nodes with the highest energy value are allowed to grow and the model reduces to *invasion percolation* [6] on a Cayley tree.

III. EDEN MODEL ON A CAYLEY TREE

Let us assume that every node has the same energy ϵ^0, i.e., $p(\epsilon)=\delta(\epsilon-\epsilon^0)$. In this case all nodes at the interface are equally likely to grow and we call this model the Eden model on a Cayley tree. The probability that a node i of the interface Int(t) grows at time t is given by $\Pi_i=1/N_{\mathrm{Int}}(t)$, where $N_{\mathrm{Int}}(t)$ is the total number of active nodes. Since at each time step a node of the interface grows, becoming part of the bulk, and m new active nodes are generated, after t time steps the model generates an interface of $N_{\mathrm{Int}}(t)$ nodes, with $N_{\mathrm{Int}}(t)=(m-1)t+1$.

We denote by $\rho(t,t_i)$ the probability that a node born at time t_i is still active at time t. Since every node grows with probability Π_i only if i is a node of the interface in the mean field, $\rho(t,t_i)$ follows

$$\frac{\partial\rho(t,t_i)}{\partial t}=-\frac{\rho(t,t_i)}{N_{\mathrm{Int}}(t)}. \tag{2}$$

Substituting $N_{\mathrm{Int}}(t)=(m-1)t+1$ in Eq. (2) in the limit $t\to\infty$ we get the solution

$$\rho(t,t_i)=\left(\frac{t_i}{t}\right)^{1/(m-1)}. \tag{3}$$

Consequently each node i that arrives at the surface at time t_i, remains at the surface with a probability that decreases in time as a power law. On the other hand the same power law describes also the distribution of the age τ of the nodes at the interface observed at time t. In fact, the probability $P(\tau)$ that a node born at time τ is still active at time t, is given by

$$P(\tau)=\left(\frac{\tau}{t}\right)^{1/(m-1)}. \tag{4}$$

Thus asymptotically in time the same power law describes the time evolution of the nodes born at time t_i, $[\rho(t,t_i)]$ and the age distribution of the nodes in the interface $[P(\tau)]$. In order to verify the theoretical predictions, we have performed numerical simulations of the Eden model on a Cayley tree with $m=2,4,6$. In Fig. 3, we report the age distribution $P(\tau)$ of the nodes at the interface for Cayley trees with $m\times 10^4$ nodes and $m=2,4,6$. The data, averaged over 100 runs, follows the power law predicted by Eq. (4). Numerical data are reported together with the theoretically predicted power law Eq. (4).

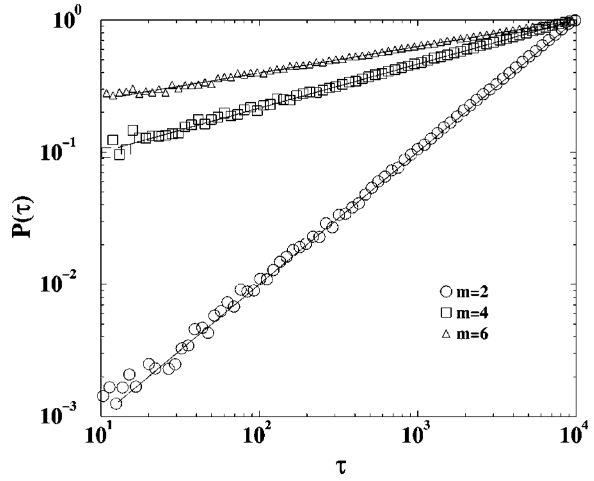

FIG. 3. Age distribution $P(\tau)$ of the nodes at the interface of a Cayley tree with connectivities $m=2,4,6$, and 10^4 generations. Data have been averaged over 100 runs. The solid lines represent the power-laws predicted by Eq. (4) with exponent $1-1/m$.

IV. CAYLEY TREE WITH ENERGIES

At finite temperature, $\beta \neq 0$ it is necessary to take into account the fact that each node has a different energy that defines its dynamics.

Since only nodes at the interface can grow, the probability that node i would leave the interface at time t is given by the product of $\rho_i(t|\epsilon_i,t_i)$ (the probability that the node is active at time t) and Π_i given by Eq. (1) (the probability that the node is chosen to grow in between the active nodes). Consequently, in the mean field, $\rho_i(t|\epsilon_i,t_i)$ decreases in time following

$$\frac{\partial \rho(t|\epsilon_{t_i},t_i)}{\partial t} = -\frac{e^{\beta\epsilon_{t_i}}\rho(t|\epsilon_i,t_i)}{\sum_j e^{\beta\epsilon_j}\rho(t|\epsilon_j,t_j)}. \tag{5}$$

In order to solve Eq. (5) we assume that in the thermodynamic limit the sum Z^S in the denominator of the left hand side of Eq. (5), given by

$$Z^S = \sum_j e^{\beta\epsilon_j}\rho(t|\epsilon_j,t_j), \tag{6}$$

self averages and converges to its mean value, approximating the sum over j with an integral over t_j

$$Z^S \to \langle Z^S \rangle = \int_0^t dt_j \int d\epsilon_j p(\epsilon_j) e^{\beta\epsilon_j}\rho(t|\epsilon_j,t_j). \tag{7}$$

Moreover, since Z^S is an extensive quantity we can self-consistently assume that $\langle Z^S \rangle$ grows linearly in time and we define the constant μ_F through

$$\langle Z^S \rangle \overset{t\to\infty}{\to} e^{\beta\mu_F} t, \tag{8}$$

Substituting this expression in the denominator sum in the dynamic Eq. (5) we found that the time evolution of $\rho_i(t|\epsilon_i,t_i)$ follows a power-law

$$\rho_i(t|\epsilon_i,t_i) = \left(\frac{t_i}{t}\right)^{f(\epsilon_i)}, \tag{9}$$

but there is multiscaling in the system, i.e., the dynamic exponent depends on the energy ϵ_i of the node,

$$f(\epsilon) = e^{\beta(\epsilon-\mu_F)}. \tag{10}$$

The probability $P(\tau)$ that a node born at time τ is still active at time t is given by a power law

$$P(\tau) = \int d\epsilon p(\epsilon)\left(\frac{\tau}{t}\right)^{e^{\beta(\epsilon-\mu_F)}} \sim \left(\frac{\tau}{t}\right)^{\delta}. \tag{11}$$

After substituting $\rho_i(t|\epsilon_i,t_i)$ from Eq. (9) with $f(\epsilon)$ given by Eq. (10), into Eq. (8), and the sum with an integral, we get the self-consistent equation for μ_F

$$1-\frac{1}{m} = \int d\epsilon p(\epsilon)\frac{1}{e^{\beta(\epsilon-\mu_F)}+1}, \tag{12}$$

formally equivalent to the definition of the chemical potential in an *equilibrium Fermi gas* suggesting that many properties of this model can be described by the Fermi statistics.

Some attention should be given to the special limits $\beta\to 0$ and $\beta\to\infty$.

$\beta\to 0$ limit. In this case we recover the solution of the Eden model on the tree, $z_F = m-1$. Since the probability distribution $p(\epsilon)$ is normalizable and the occupation number

$$n_F(\epsilon) \to \frac{1}{z_F^{-1}+1}, \tag{13}$$

Eq. (12) reduces to $z_F \to m-1$ and thus $\beta\mu_F \to \ln(m-1)$ in such a way that $\mu_F \geqslant 0$ $(z_F > 1)$ if $m > 2$.

$\beta\to\infty$ limit. In this limit the Fermi-Dirac distribution converges to the step function

$$n_F(\epsilon) \to \theta(\epsilon-\mu_F) \tag{14}$$

and the self-consistent Eq. (12) becomes

$$1-\frac{1}{m} = \int_{\epsilon<\mu_F} p(\epsilon). \tag{15}$$

In this limit and in the case of a uniform probability distribution of the energies, $p(\epsilon)=1$ and $\epsilon\in(0,1)$ the model reduces to invasion percolation on a Cayley tree, with the known result $\mu_F = 1-1/m$.

Choosing the node energy from a uniform distribution $p(\epsilon)=1$ with $\epsilon\in[0,1]$, we have simulated the growth of a Cayley tree with $m=2$ and various values of β. In Fig. 4 the distribution of the age of the nodes at the interface is shown for $\beta=2,5,10,20$ and compared to the theoretical prediction of Eq. (11), $P(\tau)\sim(\tau/t)^{\delta}$ for the uniform distribution $p(\epsilon)=1$, $\epsilon\in[0,1]$. In Fig. 5 we report the distribution of the

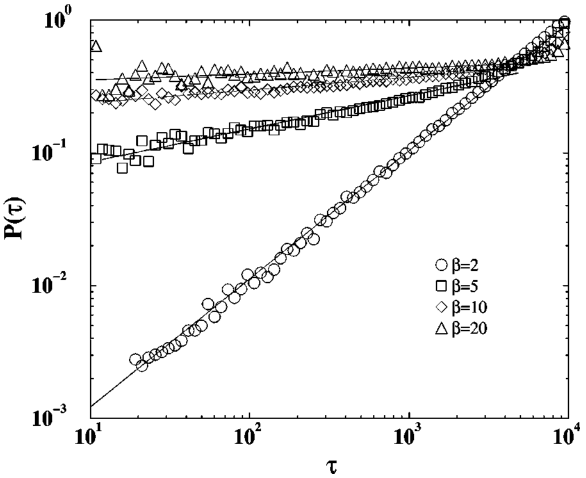

FIG. 4. Probability distribution of the age τ of the nodes at the interface in a Cayley tree with $m=2$ and time 10^4 as a function of $\beta=2,5,10,20$.

energies of the active nodes for a network of size $N=2\times10^4$ nodes for $\beta=5,10,30$. The solid line in the figure represents the theoretical prediction described by Eqs. (18) and (19) with a chemical potential given by Eq. (12).

1. Mass conservation

The self-consistent relation (12) can also be derived from mass conservation, i.e., from the knowledge that the total number of nodes at the interface is given by $N=(m-1)t$. Consequently,

$$N=(m-1)t=\sum_i \rho(t|\epsilon_i,t_i). \tag{16}$$

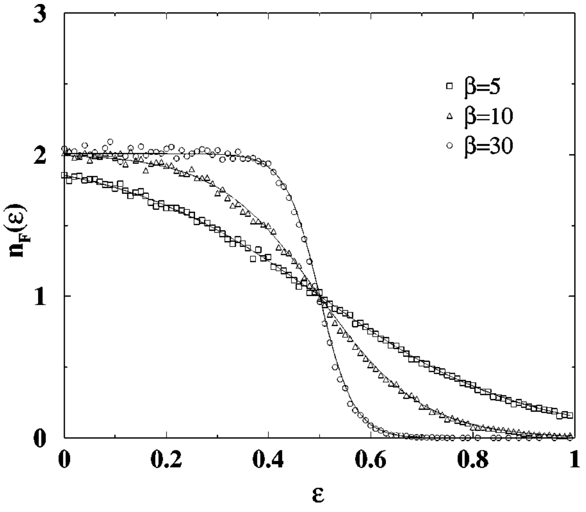

FIG. 5. The energy distribution of the nodes in the interface for $\beta=5,10,30$ in the case of a uniform energy distribution $p(\epsilon)=1$ for $\epsilon\in[0,1]$, $m=2$ and predicted chemical potential $\mu_F=1/2$. The solid lines indicate the predicted Fermi distribution.

We can substitute the sum in the right hand side of Eq. (16) with the mean over the energies ϵ_i of the nodes i of generation t_i. Moreover, in the thermodynamic limit we can approximate the sum over i with an integral over t_i, the mass conservation relation becoming

$$\begin{aligned}(m-1)t&=m\int d\epsilon p(\epsilon)\int_1^t d\tau\rho(t|\epsilon,\tau)\\ &=m\int d\epsilon p(\epsilon)\int_1^t d\tau\left(\frac{\tau}{t}\right)^{e^{\beta(\epsilon-\mu_F)}}\\ &\simeq mt\int d\epsilon p(\epsilon)\frac{1}{e^{\beta(\epsilon-\mu_F)}+1},\end{aligned} \tag{17}$$

where in the last equation we have neglected terms of order $O(t^{-\alpha})$. Thus both the mass conservation relation (16) and the self-consistent relation (12) allow us to define the chemical potential μ_F, describing the evolution of the network as the chemical potential of an equilibrium Fermi gas with specific volume $v_c=1+1/m$. However this last expression explains the meaning of that relation. In fact, the number $N_{\rm Int}(\epsilon)$ of nodes with energy ϵ at the interface at time t is given by

$$N_{\rm Int}(\epsilon)=mtn_F(\epsilon)p(\epsilon), \tag{18}$$

where $n_F(\epsilon)$ is given by the Fermi occupation number

$$n_F(\epsilon)=\frac{1}{e^{\beta(\epsilon-\mu_F)}+1}. \tag{19}$$

In other words, the distribution of the energy at the interface reaches a stationary limit given by Eq. (18) and is defined by a Fermi distribution with chemical potential given by Eq. (12). In the mean time the density of nodes with energy ϵ present in the bulk, $N_{Bulk}(\epsilon)$, reaches a stationary limit as well. In fact, since the nodes in the bulk are those of the network that are not at the interface, using Eq. (18), we have

$$N_{bulk}(\epsilon)=p(\epsilon)[1-n_F(\epsilon)]. \tag{20}$$

2. Asymptotic dynamics

The dynamical evolution of the network brings the system to the stationary state [17,18] described by the distribution function (19), as it has been shown by the solution of the dynamical Eq. (5). Moreover, the dynamics stabilizes this distribution. In fact, in the asymptotic limit, when the survivability follows Eq. (9) the probability that a node of energy ϵ will grow and leave the interface is given by

$$\pi_F(\epsilon,t)=m\int d\epsilon' p(\epsilon')\int_1^t dt'\frac{\partial\rho(t|\epsilon',t')}{\partial t}\delta(\epsilon-\epsilon'), \tag{21}$$

which can be estimated to be

$$\pi_F(\epsilon,t) = \int d\epsilon' m p(\epsilon') \int_1^t dt' \frac{e^{\beta(\epsilon-\mu_F)}\rho(t|\epsilon',t')}{t} \delta(\epsilon-\epsilon')$$

$$\simeq m p(\epsilon)[1-n_F(\epsilon)]. \tag{22}$$

Consequently, the probability that a node of energy ϵ leaves the interface, asymptotically in time, reaches a stationary limit independent of the particular evolution of the network given by

$$\pi_F(\epsilon,t) \rightarrow \pi_F^*(\epsilon) = p(\epsilon)[1-n_F(\epsilon)]. \tag{23}$$

If we observe an evolving network and we have no knowledge of the age of the nodes, but only of their energies, the complete dynamics is determined by $\pi(\epsilon,t)$ describing what is the probability that a node with energy ϵ will leave the interface at time t. While the complete dynamics (5) is clearly dependent on time, $\pi(\epsilon,t)$ reaches the stationary limit $\pi_F^*(\epsilon)$ defining the invariant dynamics of the system.

The stability of the distribution $N_{bulk}(\epsilon)$ of the energies in the bulk is thus enforced by the dynamics. In fact, we have found that, asymptotically in time, the probability that a node with energy ϵ is chosen to grow $\pi_F^*(\epsilon)$ is proportional to the number of nodes in the bulk $N_{bulk}(\epsilon)$ given by Eq. (20).

V. CONCLUSIONS

In this work we have introduced a model for a growing Cayley tree with thermal noise characterized by the following points.

(1) *Growth.* At each time exactly m nodes are added and one is eliminated at the interface, the number of nodes in which percolation can occur grows linearly in time as $N = (m-1)t$ nodes.

(2) *Time dependent dynamics.* Each node can percolate only once and the probability for a node to be chosen as the percolating one is a decreasing function of time.

These characteristics are shared with scale-free networks that are characterized by growth (new nodes being continuously added to the network) and a time dependent dynamics known as preferential attachment (nodes acquiring links in proportion to their connectivity). The stochastic model behind the construction of the two networks always involves the choice of a node in between a growing number of nodes, but while in the Cayley tree a chosen node is removed from the interface and cannot be chosen any more, in a scale-free networks there is no limit to the number of links a node can acquire. Consequently the Cayley tree is described by a Fermi distribution while the scale-free network is described by a Bose distribution.

We have solved analytically the model studying its character in the limit $\beta = 0$ and then at finite temperature. The distribution of bond strengths follows a Fermi distribution and the dynamics replicates and stabilizes this distribution, asymptotically in time. The bond strength plays the role of energy in the Fermi distribution. The distribution of ages of the node at the interface follows an effective power law. Finally this system is a symmetric construction of a power-law network following a Bose distribution, as shown in Fig. 1 and it opens the way to understand the self-organized nature of scale-free networks [1].

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